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NON-LINEAR VIBRATION PROBLEMS TREATED BY THE AVERAGING METHOD OF W.RITZ - PART I - FUNDAMENTALS OF THE METHOD

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VIBRATORY SYSTEMS, NONLINEAR ^ PHYSICS (62)
EQUATIONS, DIFFERENTIAL RADIATION (5)

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PART I

FUNDAMENTALS OF THE METHOD

BY
KARL KLOTTER

TECHNICAL REPORT NO. 17
(PART I)

PREPARED UNDER CONTRACT N6onr-251, TASK ORDER 2
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DIVISION OF ENGINEERING MECHANICS
STANFORD UNIVERSITY
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NON-LINEAR VIBRATION PROBLEMS
TREATED BY THE AVERAGING METHOD OF W. RITZ

Part I

Fundamentals of the Method

by

Karl Klotter

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Division of Engineering Mechanics

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Stanford, California

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1. Introduction: Purpose and Content of the Report.

This report on non-linear vibration problems, treated by the Averaging Method of W. Ritz, will consist of the present Part I (on the fundamentals of the method) and of several subsequent parts, which will deal with applications of the method and special aspects of the problems in question. Tentatively the contents of the following parts are:

Part II. Application of the method to systems having one degree of freedom, using a one-term-approximation. Studies of response curves.

Part III. Stability considerations.

Part IV. Application of the method to systems having one degree of freedom, using approximations of more than one term.

Part V. Application of the method to systems having more than one degree of freedom.

The investigations presented in the various parts of this report are intended to contribute to the knowledge of approximate solutions of non-linear differential equations arising from problems in the field of vibrations.

Problems of this type have received much attention in recent years. The bulk of the pertinent literature, however, is in Russian. In the USSR much work has been done in this field during the last two or three decades by a staff of highly trained scientists, which has resulted in what may be termed nearly a

Russian monopoly. It was not until very recently that the results of these vast efforts have been made available, to some extent, to the English reading public.

Three main publications are to be mentioned in this connection: N. Minorsky's report[1] [1], published in 1947, which gives an account of the essential accomplishments to be found in the Russian literature, and S. Lefschetz's translations of two outstanding books by Russian authors [2], [3]. In the course of the year 1950 two textbooks appeared in English which now facilitate an approach to the field [4], [5].

As far as some special aspects of the solutions of the pertinent differential equations are concerned (boundedness, stability, asymptotic behavior), the report of R. Bellman [6] gives a very thorough survey of the accomplishments found in the world literature.

The systems under consideration may be described by the set of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2 \cdots x_n, t), \qquad i = 1, 2, \cdots n,$$

and may be divided into two major classes: autonomous systems, in which the independent variable t does not appear explicitly on the right hand side of the equations, and non-autonomous systems, in which it does.

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1) Figures in brackets refer to the references given in section 6.

The majority of the past investigations were concerned with the autonomous systems and a number of powerful methods have been developed to deal with them. From the viewpoint of the physicist and engineer who looks for results which are readily available, understandable, and applicable, the two most important of these methods are the approximate method by Kryloff and Bogoliuboff, as described in [3], and the method of the phase plane, which is the basis of all considerations in [2]. Both methods, however, are limited to autonomous systems.

Methods dealing with non-autonomous systems, aimed at getting detailed results, are rare, and where there are such methods they are based upon either a step-by-step procedure, or an iteration procedure, thereby again reducing the system to an autonomous one. An example of the first approach is the integration method, developed and perfected very recently by L. S. Jacobsen [7], which makes use of the properties of the phase plane. An example of the second approach may be found in the method developed by M. Rauscher [8].

As powerful and important as these methods are for obtaining numerical results, they are limited by their very nature, as step-by-step or iteration methods, to special cases and do not allow one to obtain general results. What one would like to have is a method which presents the results in the closed form of an equation (or equations), thus allowing an interpretation and discussion of the influence of the parameters involved.

The purposes of the present report are to draw attention to a method which satisfies these requirements, and to present a variety of results achieved by it in dealing with the forced vibrations of non-linear systems. The method will not attempt to satisfy the given differential equation at every instant, but only in some mean average. As may already be clear from these words, this method is essentially identical with the one customarily associated with the name of B. G. Galerkin. The latter has been widely used for solving boundary value problems in statics of elastic systems. Here the method will be applied to the dynamical problems in question.

The method is also closely related to the one customarily known as Ritz's method (or as the Rayleigh-Ritz method, when applied to eigenvalue problems), and it draws much of its importance from this connection. In section 2 the two methods and their interconnection will be outlined. A closer study of the original paper of W. Ritz [9] reveals that this paper contains not only the elements but also the essential equations of the so-called Galerkin method; therefore both forms of the method must be attributed to W. Ritz. Some remarks in this connection will be found in section 3.

Section 4 will deal with additional remarks pertaining to the method. Section 5, finally, contains a summary and again -- recipe like -- a short description of the procedure for the Averaging Method.

2. The Two Forms of Ritz's Method.

a. Variational Problems and the Related Euler Equations.

Many problems in mechanics, and in other fields of physics, may be formulated as minimum problems. In each case, the solution of the problem, then, is a function which gives some integral expression a minimum value; thus the problem is described as one in the calculus of variations.

For the convenience of the reader who is not familiar with the basic concepts used in the calculus of variations, we will give here a short outline of the basic ideas as far as they are related to our purpose.

We begin with the simplest case: Let $F(x,y,y')$ be a given continuous function of the three variables x, y, y'. Now let y be a function of x, so that $y = y(x)$, and y' its derivative. Then $F(x,y,y') = F[x, y(x), y'(x)]$ becomes a function of x alone, and the integral

$$(1) \qquad I = \int_{x_0}^{x_1} F(x,y,y')\,dx$$

has a definite meaning. The numerical value of I depends of course upon

1) the boundary values x_0 and x_1,

2) the function $y(x)$, which is introduced.

Now let us consider the problem of choosing $y(x)$ in such a way as to give I a minimum value, i.e.

$$(2) \quad I = \int_{x_0}^{x_1} F(x, y, y')\, dx = \text{Minimum}$$

compared with using any other (neighboring) function y, when the boundary values x_0 and x_1 are fixed. Clearly, this problem is one of the calculus of variations.

Let us now specify the function $y(x)$ in such a way as to admit only such functions which make

$$(3) \quad y(x_0) = 0 \qquad y(x_1) = 0 .$$

The unknown true solution we denote by $\bar{y}(x)$. The arbitrary function $y(x)$ we put into the form

$$(4) \quad y(x) = \bar{y}(x) + \epsilon \eta(x)$$

where ϵ is a number and $\eta(x)$ denotes <u>any</u> allowed function [satisfying, of course, equations (3a); $\eta(x_0) = 0$, $\eta(x_1) = 0$]. Thus the integral I has become a function of the single parameter ϵ. The problem of finding the minimum of I is reduced to the ordinary problems of finding an extremum of a function of a single variable as treated in the calculus. We are looking for $\frac{dI}{d\epsilon} = 0$ while $\epsilon = 0$. Thus the following steps are clearly indicated:

$$I(\epsilon) \equiv \int_{x_0}^{x_1} F(x, y, y')\,dx = \int_{x_0}^{x_1} F(x, \bar{y}+\epsilon\eta, \bar{y}'+\epsilon\eta')\,dx$$

(5)
$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) dx = 0$$

By setting $\epsilon = 0$, the functions y and $\bar{y}$ coincide and need no longer be distinguished. Now we integrate the second term in (5) by parts:

$$\int_{x_0}^{x_1} \frac{\partial F}{\partial y'}\eta'\,dx = \left[\frac{\partial F}{\partial y'}\eta\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Because of (3a) the first term on the right hand side vanishes and we are left with

(6)
$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \eta \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] dx ,$$

which, because η is supposed to be arbitrary, calls for

(7)
$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 .$$

Equation (7) is called the "Euler Equation" of the variational problem (2). It states a necessary condition for the minimum; not a sufficient one, because it ensures only a stationary

Table 1

VARIATIONAL PROBLEMS V AND RELATED EULER EQUATIONS E

1. One dependent variable: y ; one independent variable: x ; 1st order derivatives

$$V: I = \int_{x_0}^{x_1} F(x, y, y')\, dx = \text{Minimum}$$

$$E: \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \left\| \begin{array}{l} \text{Explicitly:} \\ y''F_{y'y'} + y'F_{y'y} + F_{y'x} - F_y = 0 \end{array} \right. \quad \text{[d.e. of 2nd order]}$$

2. One dependent variable: y ; one independent variable: x ; nth order derivatives.

$$V: I = \int_{x_0}^{x_1} F(x, y, y', y'', \cdots, y^{(n)})\, dx = \text{Min}$$

$$E: \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \cdots (-1)^n \frac{d^n}{dx^n}\frac{\partial F}{\partial y^{(n)}} = 0 \quad \text{[d.e. of (2n)th order]}$$

3. More dependent variables: y , z , ---; one independent variable: x ; 1st order deriv.

$$V: I = \int_{x_0}^{x_1} F(x, y, z, \cdots, y', z', \cdots)\, dx = \text{Min}$$

$$E: \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \quad \left\| \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0 \quad \right\| \quad \text{-------}$$

4. For higher order derivatives, equations are like the ones under #2.

5. One dependent variable: u ; two independent variables: x, y ; 1st order deriv.

$$V: I = \iint_{(R)} F(x, y, u, u_x, u_y)\, dx\, dy = \min$$

$$E: \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} = 0$$

6. Further generalizations are obvious from these examples.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial U}{\partial y} = 0 \tag{9}$$

which is exactly Langrange's equation.

There are, therefore, two equivalent ways of stating a statical or dynamical problem:

1) as a variational problem,

2) by its differential equation, together with the boundary values.

b. Ritz's First Procedure of Approximating the Solution.

Again, we limit our attention to a system of a single degree of freedom. The variational problem in question may, e.g., be given by

$$I = \int_{x_0}^{x_1} F(x, y, y' - - - y^{(n)})\, dx \equiv \text{Minimum} \tag{10}$$

corresponding to case 2 of Table 1. As we have said before, I is a function of the (required) function $y(x)$ and it depends on its functional infinity of values in the interval (x_0, x_1). Ritz's idea was to make I depend on a finite number of parameters only, by replacing (or approximating) $y(x)$ by a function $\tilde{y}(x)$,

$$(11) \qquad y(x) = a_1 \psi_1(x) + a_2 \psi_2(x) + \cdots + a_n \psi_n(x)$$

where the $\psi_k(x)$ are a given set of functions and the a_k are constants to be determined. The original problem is now reduced to the problem of giving the integral I, which depends on a finite number of parameters a_k, a minimum value. Clearly, this is now an ordinary maximum-minimum problem of a function of several variables. The necessary conditions (ensuring a stationary value for I) become

$$(12) \qquad \frac{\partial I}{\partial a_1} = 0; \quad \frac{\partial I}{\partial a_2} = 0; \quad \cdots; \quad \frac{\partial I}{\partial a_n} = 0.$$

These are n equations, from which the n unknown parameters $a_1, a_2, \cdots a_n$ can be determined.

In all cases, where the differential equation (Euler equation) is linear, the integral I is a quadratic function in y and its derivatives. Therefore, the conditions (12) reduce to a system of linear algebraic equations for the $a_1 \cdots a_n$. In those cases, where the differential equation is non-linear, the equations (12) also are non-linear algebraic equations.

The conditions (12) ensure that we get the "best" solution [under the restriction (11)], because we make the integral I have a minimum (or at least a stationary) value. The question of how

much this minimum differs from the (lowest) minimum, connected with the true solution, remains open. All we know is that we get the "best" solution (lowest value for I) obtainable with a function of the form (11).

W. Ritz [9] has proved that $\tilde{y}$ converges into the true solution y with increasing number of terms in (11), provided that the set of functions ψ_k is a complete set. E. Trefftz [10] subsequently investigated again the conditions for convergence and in addition the error limits of the method.

In pure mathematical terms one can say that by assuming the solution to have the form of (11) one reduces the parameter space for the solution from an infinite number of dimensions to a finite number of dimensions; or, one projects the solution into such a parameter space of a finite number of dimensions. The accuracy of the results, of course, will depend largely on whether or not the essential features of the solution are preserved by doing so.

Clearly, the accuracy will depend upon the number of terms used in (11) and especially on the choice of an appropriate set of functions ψ_k . As one writer ([13b] p. 493) terms it: "[There is] opportunity for the display of skill in the choice of the functions $\psi_k(x)$ ".

Obviously, some advance knowledge regarding the expected solution will greatly facilitate the choice of the coordinate functions ψ_k . By making a good choice it is often possible to restrict the assumption (11) to a single term. Part II of this

report will deal exclusively with such single term approximations.

c. A Second, Equivalent, Procedure for the Approximation.

In applying the conditions (12) to the integral (10) where y is replaced by $\tilde{y}$ according to the assumption (11), we get

$$(13)\qquad \frac{\partial I}{\partial a_k} = \int_{x_0}^{x_1} \left[\frac{\partial F}{\partial \tilde{y}} \psi_k + \frac{\partial F}{\partial \tilde{y}'} \psi_k' + \cdots + \frac{\partial F}{\partial \tilde{y}^{(n)}} \psi_k^{(n)} \right] dx$$

and by integrating the second and the subsequent terms by parts we arrived at

$$\frac{\partial I}{\partial a_k} = \left[\psi_k \frac{\partial F}{\partial \tilde{y}'} \right] - \left[\psi_k' \frac{\partial F}{\partial \tilde{y}''} - \psi_k \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{y}''} \right) \right] + \cdots$$

$$(14)\qquad + \int_{x_0}^{x_1} \left\{ \frac{\partial F}{\partial \tilde{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{y}'} \right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial \tilde{y}''} \right) - + \cdots (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial \tilde{y}^{(n)}} \right) \right\} dx$$

By comparison with the formulae in Table 1 we see that the braces in the last integral contain the expression E for the Euler equation of the problem; so, we are left with

$$(15)\qquad \frac{\partial I}{\partial a_k} = \left[\text{integrated parts} \right]_{x_0}^{x_1} + \int_{x_0}^{x_1} \psi_k \, E[\tilde{y}] \, dx$$

The requirements $\frac{\partial I}{\partial a_k} = 0$ therefore call for the following conditions:

1) the integrated parts must vanish;

$$2) \quad \int_{x_0}^{x_1} \psi_k \, E[\tilde{y}] \, dx = 0$$

The integrated parts vanish if either ψ_k and its derivatives up to the nth order vanish or the corresponding terms $\frac{\partial F}{\partial \tilde{y}'}$, $\frac{\partial F}{\partial \tilde{y}''}$, $\frac{d}{dx}\left(\frac{\partial F}{\partial \tilde{y}''}\right)$, ---- vanish, at the boundaries x_0 and x_1. Let us assume that the integrated parts do vanish (since it can always be accomplished by a proper choice of the functions ψ_k). Then the conditions (12) become <u>equivalent</u> to the set of equations

$$(16) \quad \int_{x_0}^{x_1} \psi_k \, E[\tilde{y}] \, dx = 0, \qquad k = 1, 2, \text{---}, n.$$

$E[y]$ is the differential equation of the problem, and it coincides with the Euler equation of the corresponding variational problem; however, $E[\tilde{y}]$ will not vanish at every point. The equations (16) tell us that the best solution $\tilde{y}$ will be that which makes $E[\tilde{y}]$ vanish in some <u>"weighted" average</u>. The weight functions thereby coincide with the coordinate functions of assumption (11).

As is clear from this analysis the conditions (16) are equivalent to (12) (provided the integrated parts vanish). This means that satisfying the differential equation $E[\tilde{y}]$ in the described weighted average ("Averaging Method") leads to the same result

of minimizing the integral I as do the equations (12) themselves ("Minimizing Method"). In short: the Minimizing Method (12) is equivalent to the Averaging Method (16). Equations (16) have the considerable advantage of making use of the differential equation itself and not of the expression I; one need not even know the expression I, in case the problem is stated by giving the differential equation.

Of course, all the remarks about convergence, degree of approximations, choice of coordinate functions (and weight functions) ψ_k, etc., from the end of section 2b, apply here in just the same way.

3. Remarks on Literature and on Historical Developments.

We have seen that there are two ways of determining the coefficients a_k in the assumed form of an approximate solution (11): either by introducing (11) into the corresponding integral expression (10), which is to be minimized, thus leading to the equations (12) for the a_k; or by introducing the assumption (11) into the differential equation of the problem (which is the Euler equation of the corresponding variational problem) and averaging this expression, using the coordinate functions ψ_k as weight functions, according to (16). If the boundary conditions (arising from the integrated parts) are satisfied, both procedures lead to identical results; in fact, they are identical and represent just two views of the same problem.

It has become customary to call the first procedure, which results in equations (12), the Ritz Method (or Rayleigh-Ritz method, when applied to eigenvalue problems), referring to the paper [9] of 1909; and the second procedure, the Galerkin Method, referring to the paper [11] of 1915.

The interdependency and equivalence of the two methods was noted and described several times. It may be sufficient to mention the three sources [12], [13], and [14]. In [12] the equivalence is shown for the boundary value problems in statics of elastic systems and for eigenvalue problems; in [13b] for eigenvalue problems also; and in [14] in a most general manner.

Curiously, however, of all these writers who have shown the interconnection of "Ritz's" and "Galerkin's" method, none was aware of the fact that in Ritz's original paper [9] he himself made use of both procedures; and not only by inference, but explicitly he states the basic equation [see equation (41) in [9]], corresponding to (16), for the special problem (bending of a plate) which he treats there.

On the other hand, it must be admitted that Ritz does not stress this point (of having two methods) and does not draw special attention to it. Therefore, apparently most of the readers of Ritz's paper missed the importance of the statement in eq. (41) of this paper and had to be told of the second method by B. G. Galerkin, who -- without referring to Ritz -- used this method for solving similar problems for rods and plates. Inasmuch as

Ritz himself does state the important equation in his paper of 1909, and since there is little doubt that [because of the close scientific connections in those days between St. Petersburg, Russia, the domicile of Galerkin, and Göttingen, the place where Ritz lived and wrote his paper],[1] Galerkin had full knowledge of the Ritz paper; both methods therefore must be ascribed to Ritz, in the interest of historical accuracy. The proper way of referring to them would be to call them the First and the Second Ritz Method, respectively; or, if one wishes to distinguish between the methods in the title they may be termed, "Ritz's Minimizing Method" and "Ritz's Averaging Method". Labelling one as Galerkin's Method may be permissible only as a "manner of speaking" and as a full equivalent to the "Second Ritz Method" or "Ritz's Averaging Method".

The fact that the fame of Galerkin's paper [11] of 1915 has spread so much, and has even obscured the statements in Ritz's paper of 1909, is even more astonishing because the majority of the subsequent writers were unable to read the paper [11] written in Russian, and sometimes were even unable to obtain it (see the introductory remark in [13a]). The scientific world outside Russia apparently learned of Galerkin's paper and the method used therein from two later papers: first, from H. Hencky's paper [16] and the remarks by which C. B. Biezeno drew special attention

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1) (S. P. Timoshenko, a close colleague of B. G. Galerkin, visited Göttingen frequently and promoted discussion of the Ritz method in St. Petersburg; see, e.g., his paper [15] of 1913).

to it at the First International Congress for Applied Mechanics, Delft 1924 [17]; and second, according to [13a], from the paper by E. P. Grossmann [18].

4. Additional Remarks Concerning the Averaging Method.

In section 2 it was pointed out that the Averaging Method is equivalent to the Minimizing Method. The advantage of the Averaging Method is based upon two facts. 1) In case the problem is stated by the differential equation, the Averaging Method allows one to dispense with the integral expression I completely, which is a considerable advantage, especially if this expression is not known and would have to be determined for the purpose. 2) Even if I is available, the Averaging Method normally presents fewer difficulties in calculating the expressions involved; it is usually the easier method from the point of view of practical calculations.

The advantages, however, are still greater, inasmuch as the method allows generalizations. Such generalizations [1] were suggested in three directions.

First, the method has a meaning of its own in cases where an expression I does not even exist. Then the relationship to some minimum value ceases to exist, but the averaging process may be carried out nevertheless.

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1) Some of these suggestions are mentioned in lecture notes by K. O. Friedrichs [19].

Second, whereas the relationship to the Minimizing Method requires the weight functions in the Averaging Method to be identical with the coordinate functions in the assumption, this relationship may be dropped, and one might try to solve a problem by using weight functions which differ from the coordinate functions.

Third, in (11) the approximating function $\tilde{y}$ was assumed to be a polynomial with coefficients to be determined. Cases may exist where other assumptions, say

$$\tilde{y} = A e^{\alpha x} \tag{17}$$

(with A and α as parameters to be determined), or the like, may prove advantageous.

The foregoing suggestions, of course, have to be carried out with some discretion and one must be aware of possible failures. The merits of the results will have to be judged by the success.

Nowhere in this report, however, will the suggestions of the generalized procedure be applied.

5. Summary.

1. It can be shown, generally and without reference to any particular type of problem (eigenvalue problems, statics in elastic bodies, etc.) that the "Minimizing Method" and the "Averaging Method" (with a proper choice of the coordinate functions) are equivalent.

2. Whereas the Minimizing Method is generally attributed to W. Ritz (or to Lord Rayleigh and W. Ritz, if eigenvalue problems are involved,) the Averaging Method is customarily attributed to B. G. Galerkin. However, Ritz's paper contains the essential averaging equation explicitly; for reasons of historical accuracy both methods are to be ascribed to W. Ritz.

3. The Averaging Method has some advantages over the Minimizing Method. They lie 1) in the ease of computations, and 2) in the fact that for applying it one needs to know only the differential equation, irrespective of whether or not a variational expression I is known, or even exists.

Both of these reasons recommend the Averaging Method for exclusive and wide use in the subsequent parts of this report.

4. The procedure for applying the Averaging Method is extremely simple. Let $E[y(x)] = 0$ be the differential equation (linear or not) of the problem (autonomous or not); then the required solution $y(x)$ is approximated by

$$\tilde{y}(x) = a_1 \psi_1(x) + a_2 \psi_2(x) + \cdots + a_n \psi_n(x) ,$$

where the $\psi_k(x)$ are a given set of functions. The "best" approximation has coefficients a_k which are determined by the n equations

$$\int_{x_0}^{x_1} E[\tilde{y}(x)]\, \psi_k(x)\, dx = 0 \qquad k = 1, 2, \cdots n.$$

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